SUPPLEMENTAL MATERIALS

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Seismic Response of Inhomogeneous Soil Deposits with Exponentially Varying Stiffness

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Solution to Eq. 4

Despite the simplification over the original governing equation in Eq. 3, Eq. 4 is still intractable because of the transcendental nature of the variable wavenumber $k(z)$. To overcome this difficulty, one may introduce the non-linear coordinate transformation

$$x = e^{-2\alpha\, z/z_r} \tag{S1}$$

in which x is a new dimensionless independent variable.

Substituting Eq. S1 back to Eq. 4 and employing the chain rule of differentiation, Eq. 4 simplifies to

$$x^2 \frac{d^2}{dx^2} Y + x \frac{d}{dx} Y + \left(\frac{k_0\, z_r}{2\alpha}\right)^2 x\, Y = 0 \tag{S2}$$

in which the first derivative of the dependent variable, dY/dx, has re-appeared, yet the variable coefficients are now simple monomials of x.

Following Wylie & Barrett (1986) and Trachanas (2004), a simple monomial solution in the form $Y(x) = x^s$ is tried, which is assumed valid both at small and large values of the argument x. Substituting this trial solution into Eq. S2 yields the indicial equation

$$[s\,(s-1) + s]\, x^s + \left(\frac{k_0\, z_r}{2\alpha}\right)^2 x^{s+1} = 0 \tag{S3}$$

Evidently, for small values of x the first term in Eq. S3, associated with the lowest exponent, s, dominates, which provides two non-trivial solutions if $s = 0$ (double root). Accordingly, $x = 0$ is a regular singular point and the solution can be expressed locally as a power-law function of x. An alternative proof of this can be established in light of Fuch's theorem, the conditions of which are satisfied by Eq. S2 at $x = 0$. On the other hand, for large values of x, the second term in Eq.

S3, associated with the largest exponent, s+1, dominates, which has no solutions in terms of s. This suggests that infinity is an irregular singular point, and the solution does not behave locally as a power function of x.

In light of the above properties, Eq. S2 is of the Bessel type and admits the general solution (Wylie and Barrett, 1986; Trachanas, 2004):

$$Y(x) = x^{\mu}[C_1\, J_v(\lambda x^{l/2}) + C_2\, N_v(\lambda x^{l/2})] \tag{S4}$$

where $J_v($ $)$ and $N_v($ $)$ are the Bessel functions of the first and the second kind and order v, respectively, while C_1, C_2 are integration constants to be determined from the boundary conditions. In the solution at hand,

- l = 1 is a dimensionless parameter representing the step of the associated Frobenious power series solutions near the origin, and is equal to the difference between the powers of x in Eq. S3 i.e. (s+1) – (s) = 1.
- μ = 0 is the exponent of the monomial multiplier in Eq. S4 and is equal to the average of the two values of s (both equal to zero) in the polynomial solution in Eq. S3.
- v = 0 is the order of the Bessel functions which is equal to the difference between the values of s in Eq. S3 divided by the step of the power series, l.
- Parameter λ is associated with the asymptotic behavior of the solution at infinity and can be determined by substituting the following asymptotic relations into Eq. S2

$$Y_{\infty} \sim e^{\pm i\lambda x^{l/2}}, \quad Y'_{\infty} \sim \pm\frac{1}{2} i\, \lambda\, l\, x^{l/2-1}\, e^{\pm i\lambda x^{l/2}}, \quad Y''_{\infty} \sim \mp\frac{1}{4}\lambda^2\, l^2\, x^{l-2}\, e^{\pm i\lambda x^{l/2}} \tag{S5}$$

to get

$$\lambda = \pm\frac{k_0 z_r}{\alpha} \tag{S6}$$

in which both signs are admissible.

In light of the above, the solution in Equation S4 can be written in the explicit form:

$$Y(x) = C_1 \, J_0 \left(\pm \frac{k_0 z_r}{\alpha} x^{1/2} \right) + C_2 \, N_0 \left(\pm \frac{k_0 z_r}{\alpha} x^{1/2} \right) \tag{S7}$$

Switching back to the original independent variable *z* and retaining the positive sign in the argument of the Bessel functions, the solution can be cast in the form of Eq. 6.

Proportionality relation between C_1 and C_2

Imposing the boundary condition of a traction-free surface, τ(0) = 0, Eq. 8 yields a proportionality relation between C_1 and C_2 :

$$C_2 = - \, C_1 \, J_0 \left(\frac{k_0 z_r}{\alpha} \right) / N_0 \left(\frac{k_0 z_r}{\alpha} \right) \tag{S8}$$

Displacement at soil surface *u*(0)

For z = 0, the displacement at soil surface is obtained as

$$u(0) = C_1 k_0 \left[J_1 \left(\frac{k_0 z_r}{\alpha} \right) N_0 \left(\frac{k_0 z_r}{\alpha} \right) - J_0 \left(\frac{k_0 z_r}{\alpha} \right) N_1 \left(\frac{k_0 z_r}{\alpha} \right) \right] / N_0 \left(\frac{k_0 z_r}{\alpha} \right) \tag{S9}$$

Employing Lommel's identity (Abramowitz & Stegun, 1965)

$$J_1 \left(\frac{k_0 z_r}{\alpha} \right) N_0 \left(\frac{k_0 z_r}{\alpha} \right) - J_0 \left(\frac{k_0 z_r}{\alpha} \right) N_1 \left(\frac{k_0 z_r}{\alpha} \right) = (2\alpha / \pi k_0 z_r) \tag{S10}$$

the expression for displacement at z = 0 simplifies to that in Eq. 13.

Derivation of Rayleigh Solutions

To derive the fundamental frequency of the system by means of the Rayleigh quotient, both terms in Eq. 3 are multiplied by an arbitrary trial function of depth, $u^*(z)$, that satisfies the displacement (i.e. essential) boundary conditions of the problem in the same manner as the unknown function $u(z)$

$$\frac{d}{dz}\left[G(z)\frac{du(z)}{dz}\right]u^*(z) = -\omega^2\,\rho(z)\,u(z)\,u^*(z) \tag{S11}$$

Upon integrating over depth and employing integration by parts in the first term of Eq. S11, one obtains:

$$G(z)\frac{du(z)}{dz}u^*(z)\Big|_0^H - \int_0^H G(z)\frac{du(z)}{dz}\frac{du^*(z)}{dz}dz = -\omega^2\int_0^H \rho(z)u(z)u^*(z)dz \tag{S12}$$

The first term on the left-hand side of Eq. S12 expresses the difference between the product of actual stresses and virtual displacements at the two ends of the medium. Both these terms are zero due to the presence of the stress-free ground surface [i.e., $\tau(0)$ = 0] and the fixed base [i.e., $u^*(H)$ = 0]. It should be noted that the first of these conditions is enforced although the selected function $u(z)$ does not necessarily satisfy $du(0)/dz = 0$.

Further, assuming $u^*(z) = u(z)$, Eq. S12 simplifies to:

$$\int_0^H G(z)\left(\frac{du(z)}{dz}\right)^2 dz = \omega^2\int_0^H \rho(z)\,u^2(z)\,dz \tag{S13}$$

which leads directly to Eq. 19.

To determine the compatible mode shape in Table 1, the inhomogeneous soil column is loaded horizontally by a static distributed load equal to the soil unit weight at each elevation. This leads to the following ordinary differential equation

$$\frac{d}{dz}\left[G(z)\frac{du}{dz}\right] = \rho(z)\,g \tag{S14}$$

For constant mass density, the slope of the column displacement is obtained by integration of Eq. S14 over z

$$G(z)\frac{du}{dz} = \rho\, g\, z + c_1 \tag{S15}$$

where c_1 is an integration constant. Since the stiffness $G(z)$ is finite at the surface, the corresponding shear strain, $\gamma(0)$, is zero which requires $c_1 = 0$.

Using Eq. 1 and carrying out a second integration with depth, one obtains after simplification

$$u(z) = c_2 - \left(\frac{\rho g}{G_0}\right)\left(\frac{H}{2a}\right)\left[z + \left(\frac{H}{2a}\right)\right] e^{-2az/H} \tag{S16}$$

Imposing the boundary condition of zero displacement at the base, one obtains the displacement function

$$u(z) = \left(\frac{\rho\, g}{G_0}\right)\left(\frac{H}{2a}\right)^2 \left\{\left(2a\frac{z}{H} + 1\right)\, e^{-2az/H} - (2a+1)\, e^{-2a}\right\} \tag{S17}$$

from which the displacement at the top of the layer is readily calculated as

$$u(0) = \left(\frac{\rho\, g}{G_0}\right)\left(\frac{H}{2a}\right)^2 \{1 - (2a+1)\, e^{-2a}\} \tag{S18}$$

Normalising Eq. S17 by $u(0)$ in Eq. S18 and setting $\eta = z/H$, yields the solution of the "compatible" shape function in Table 1.

Asymptotic formulae of the Bessel functions for large arguments

The asymptotic behaviour of the relevant Bessel functions for large arguments is (Abramowitz and Stegun, 1965)

$$J_0(x) \sim \left(\frac{2}{\pi\, x}\right)^{1/2} cos\left(x - \frac{\pi}{4}\right) \tag{S19}$$

$$N_0(x) \sim \left(\frac{2}{\pi\, x}\right)^{1/2} sin\left(x - \frac{\pi}{4}\right) \tag{S20}$$

$$J_1(x) \sim \left(\frac{2}{\pi\, x}\right)^{1/2} cos\left(x - \frac{3\pi}{4}\right) \tag{S21}$$

$$N_1(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} sin\left(x - \frac{3\pi}{4}\right) \tag{S22}$$

These approximations hold for real arguments $x >> 3/4$ for the functions of the first order, and real arguments $x >> 1/4$ for the functions of the zero order.

Asymptotic solutions in the high-frequency regime

Based on the expressions S19 to S22, in the high-frequency regime, the following asymptotic expressions for the displacement, shear strain, curvature and shear stress profiles with depth in Eqs. 9, 10, 12 and 8 are possible:

$$u(z) = C_1 \left(\frac{2\alpha}{\pi H}\right)^{1/2} k_0^{1/2}\, e^{-\alpha\frac{z}{2H}}\, cos\left[\frac{k_0\, H}{\alpha}\left(1 - e^{-\alpha\frac{z}{H}}\right)\right] / sin\left[\frac{k_0\, H}{\alpha} - \frac{\pi}{4}\right] \tag{S23}$$

$$\gamma(z) = -C_1 \left(\frac{2\alpha}{\pi H}\right)^{1/2} k_0^{3/2}\, e^{-3\alpha\frac{z}{2H}}\, sin\left[\frac{k_0\, H}{\alpha}\left(1 - e^{-\alpha\frac{z}{H}}\right)\right] / sin\left[\frac{k_0\, H}{\alpha} - \frac{\pi}{4}\right] \tag{S24}$$

$$(1/R) = C_1 \left(\frac{2\alpha}{\pi\, H}\right)^{1/2} k_0^{3/2}\, e^{-3\alpha\frac{z}{2H}} \left\{\frac{\frac{2\alpha}{H}\, sin\left[\frac{k_0\, H}{\alpha}\left(1-e^{-\alpha\frac{z}{H}}\right)\right] - k_0 e^{-\alpha\frac{z}{H}}\, cos\left[\frac{k_0\, H}{\alpha}\left(1-e^{-\alpha\frac{z}{H}}\right)\right]}{sin\left[\frac{k_0 H}{\alpha} - \frac{\pi}{4}\right]}\right\} \tag{S25}$$

$$\tau(z) = -C_1 \rho\omega^2 \left(\frac{2\alpha}{\pi\, H}\right)^{1/2} k_0^{-1/2}\, e^{\alpha\frac{z}{2H}}\, sin\left[\frac{k_0\, H}{\alpha}\left(1 - e^{-\alpha\frac{z}{H}}\right)\right] / sin\left[\frac{k_0\, H}{\alpha} - \frac{\pi}{4}\right] \tag{S26}$$

Transfer Matrix Formulation

For an inhomogeneous layer of thickness h_i, shear modulus G_i, shear wave propagation velocity V_{0i}, wavenumber at the origin $k_{0i} = \omega/V_{0i}$, and inhomogeneity parameter α_i, upon implementing Eqs. 7 and 8 and setting $z_r = h_i$, the state vector expressing displacements and shear stresses as functions of the undetermined coefficients C_1 and C_2, is:

$$[S(z)]_i = \begin{bmatrix} u(z) \\ \tau(z) \end{bmatrix} = \begin{bmatrix} k_{0i} e^{-\alpha_i \frac{z}{h_i}}\, J_1\left(\frac{k_{0i} h_i}{\alpha_i} e^{-\alpha_i \frac{z}{h_i}}\right) & k_{0i} e^{-\alpha_i \frac{z}{h_i}}\, N_1\left(\frac{k_{0i} h_i}{\alpha_i} e^{-\alpha_i \frac{z}{h_i}}\right) \\ -\rho\omega^2 J_0\left(\frac{k_{0i} h_i}{\alpha_i} e^{-\alpha_i \frac{z}{h_i}}\right) & -\rho\omega^2 N_0\left(\frac{k_{0i} h_i}{\alpha_i} e^{-\alpha_i \frac{z}{h_i}}\right) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = [\Delta(z)]_i \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

(S27)

The transfer matrix [L] relating the state vector at the top and the bottom of the layer can be obtained in a straightforward manner as

$$[L]_i = [\Delta(h)]_i[\Delta(0)]_i^{-1} \tag{S28}$$

which yields the explicit solution

$$[L]_i = \\ = \frac{1}{A}\begin{bmatrix} e^{-\alpha_i}\left[J_1\left(\frac{k_{0i}h_i}{\alpha_i}e^{-\alpha_i}\right)N_0\left(\frac{k_{0i}h_i}{\alpha_i}\right) - J_0\left(\frac{k_{0i}h_i}{\alpha_i}\right)N_1\left(\frac{k_{0i}h_i}{\alpha_i}e^{-\alpha_i}\right)\right] & \frac{e^{-\alpha_i}k_{0i}}{\rho\omega^2}\left[J_1\left(\frac{k_{0i}h_i}{\alpha_i}e^{-\alpha_i}\right)N_1\left(\frac{k_{0i}h_i}{\alpha_i}\right) - J_1\left(\frac{k_{0i}h_i}{\alpha_i}\right)N_1\left(\frac{k_{0i}h_i}{\alpha_i}e^{-\alpha_i}\right)\right] \\ \frac{\rho\omega^2}{k_{0i}}\left[J_0\left(\frac{k_{0i}h_i}{\alpha_i}\right)N_0\left(\frac{k_{0i}h_i}{\alpha_i}e^{-\alpha_i}\right) - J_0\left(\frac{k_{0i}h_i}{\alpha_i}e^{-\alpha_i}\right)N_0\left(\frac{k_{0i}h_i}{\alpha_i}\right)\right] & J_1\left(\frac{k_{0i}h_i}{\alpha_i}\right)N_0\left(\frac{k_{0i}h_i}{\alpha_i}e^{-\alpha_i}\right) - J_0\left(\frac{k_{0i}h_i}{\alpha_i}e^{-\alpha_i}\right)N_1\left(\frac{k_{0i}h_i}{\alpha_i}\right) \end{bmatrix}$$

(S29)

in which

$$A = J_1\left(\frac{k_{0i}h_i}{\alpha_i}\right)N_0\left(\frac{k_{0i}h_i}{\alpha_i}\right) - J_0\left(\frac{k_{0i}h_i}{\alpha_i}\right)N_1\left(\frac{k_{0i}h_i}{\alpha_i}\right) \tag{S30}$$

If the above transfer matrix is combined with the familiar counterpart for a homogeneous layer (Mylonakis 1995)

$$[L]_i = [\Delta(h)]_i[\Delta(0)]_i^{-1} = \begin{bmatrix} \cos(k_ih_i) & \frac{1}{k_iG_i}\sin(k_ih_i) \\ -G_ik_i\sin(k_ih_i) & \cos(k_ih_i) \end{bmatrix} \tag{S31}$$

multi-layer systems involving an arbitrary number of stacked layers of constant stiffness and exponentially varying stiffness with depth can be handled.

For example, consider an inhomogeneous layer of thickness h_1, shear wave propagation velocity at surface V_{01}, wavenumber at the origin $k_{01} = \omega/V_{01}$, and soil mass density ρ_1 over a homogeneous layer of thickness h_2, shear wave propagation velocity V_2, shear modulus G_2, wavenumber $k_2 = \omega/V_2$ and soil mass density ρ_2.

The state vector $[S(h_2)]_2$ at the bottom of the homogeneous layer is related to $[S(0)]_2$, referring to the state vector at the top of the layer, through the transfer matrix $[L]_2$:

$$[S(h_2)]_2 = [L]_2[S(0)]_2 \tag{S32}$$

Continuity of stresses and displacements at the interface between the two layers requires:

$$[S(0)]_2 = [S(h_1)]_1 \tag{S33}$$

where,

$$[S(h_1)]_1 = [L]_1[S(0)]_1 \tag{S34}$$

Replacing Eqs. (S33) and (S34) in Eq. (S32) and setting [D]=$[L]_2[L]_1$ yields:

$$[S(h_2)]_2 = [D][S(0)]_1 \tag{S35}$$

Upon implementing Eqs. (S29) and (S31), matrix [D] is equal to:

$$[D] = \frac{1}{A}\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \tag{S36}$$

where

$$D_{11} = cos(k_2h_2)e^{-\alpha}\left[J_1\left(\frac{k_{01}h_1}{\alpha}e^{-\alpha}\right)N_0\left(\frac{k_{01}h_1}{\alpha}\right) - J_0\left(\frac{k_{01}h_1}{\alpha}\right)N_1\left(\frac{k_{01}h_1}{\alpha}e^{-\alpha}\right)\right] + \frac{\rho_1\omega^2 sin(k_2h_2)}{k_{01}k_2h_2}\left[J_0\left(\frac{k_{01}h_1}{\alpha}\right)N_0\left(\frac{k_{01}h_1}{\alpha}e^{-\alpha}\right) - J_0\left(\frac{k_{01}h_1}{\alpha}e^{-\alpha}\right)N_0\left(\frac{k_{01}h_1}{\alpha}\right)\right]$$

$$D_{12} = \frac{k_{01}cos(k_2h_2)}{\rho_1\omega^2}e^{-\alpha}\left[J_1\left(\frac{k_{01}h_1}{\alpha}e^{-\alpha}\right)N_1\left(\frac{k_{01}h_1}{\alpha}\right) - J_1\left(\frac{k_{01}h_1}{\alpha}\right)N_1\left(\frac{k_{01}h_1}{\alpha}e^{-\alpha}\right)\right] + \frac{sin(k_2h_2)}{k_2h_2}\left[J_1\left(\frac{k_{01}h_1}{\alpha}\right)N_0\left(\frac{k_{01}h_1}{\alpha}e^{-\alpha}\right) - J_0\left(\frac{k_{01}h_1}{\alpha}e^{-\alpha}\right)N_1\left(\frac{k_{01}h_1}{\alpha}\right)\right]$$

$$D_{21} = -G_2k_2sin(k_2h_2)e^{-\alpha}\left[J_1\left(\frac{k_{01}h_1}{\alpha}e^{-\alpha}\right)N_0\left(\frac{k_{01}h_1}{\alpha}\right) - J_0\left(\frac{k_{01}h_1}{\alpha}\right)N_1\left(\frac{k_{01}h_1}{\alpha}e^{-\alpha}\right)\right] + \frac{\rho_1\omega^2 cos(k_2h_2)}{k_{01}}\left[J_0\left(\frac{k_{01}h_1}{\alpha}\right)N_0\left(\frac{k_{01}h_1}{\alpha}e^{-\alpha}\right) - J_0\left(\frac{k_{01}h_1}{\alpha}e^{-\alpha}\right)N_0\left(\frac{k_{01}h_1}{\alpha}\right)\right]$$

$$D_{22} = -\frac{G_2k_2sin(k_2h_2)k_{01}e^{-\alpha}}{\rho_1\omega^2}\left[J_1\left(\frac{k_{01}h_1}{\alpha}e^{-\alpha}\right)N_1\left(\frac{k_{01}h_1}{\alpha}\right) - J_1\left(\frac{k_{01}h_1}{\alpha}\right)N_1\left(\frac{k_{01}h_1}{\alpha}e^{-\alpha}\right)\right] + cos(k_2h_2)\left[J_1\left(\frac{k_{01}h_1}{\alpha}\right)N_0\left(\frac{k_{01}h_1}{\alpha}e^{-\alpha}\right) - J_0\left(\frac{k_{01}h_1}{\alpha}e^{-\alpha}\right)N_1\left(\frac{k_{01}h_1}{\alpha}\right)\right]$$

FIGURES

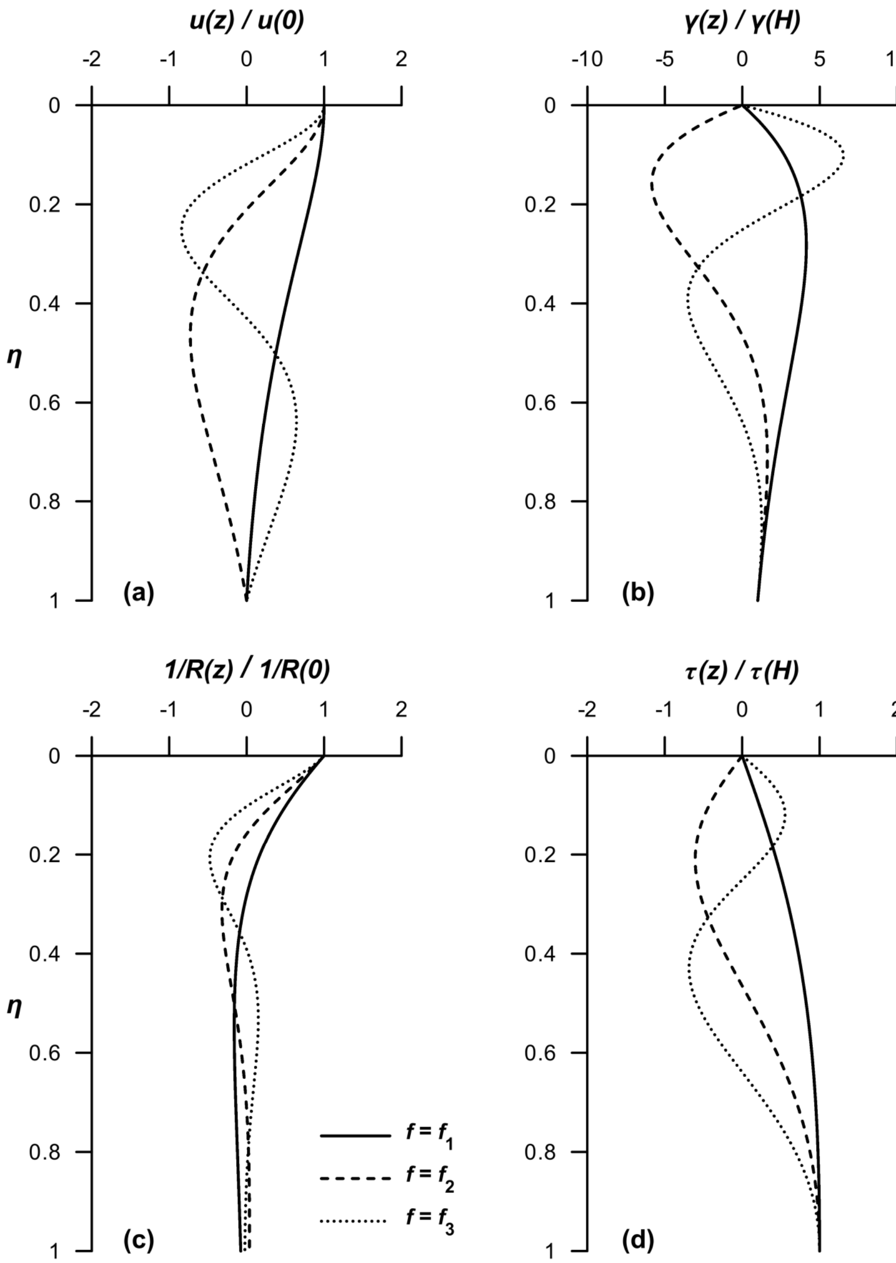

Figure S1: Distribution of normalized (a) displacements, (b) shear strains, (c) curvatures and (d) shear stresses with depth of an inhomogeneous layer with $V_0/V_H = 0.25$ and excitation frequency equal to the resonant frequencies of the soil; $\xi = 0.05$

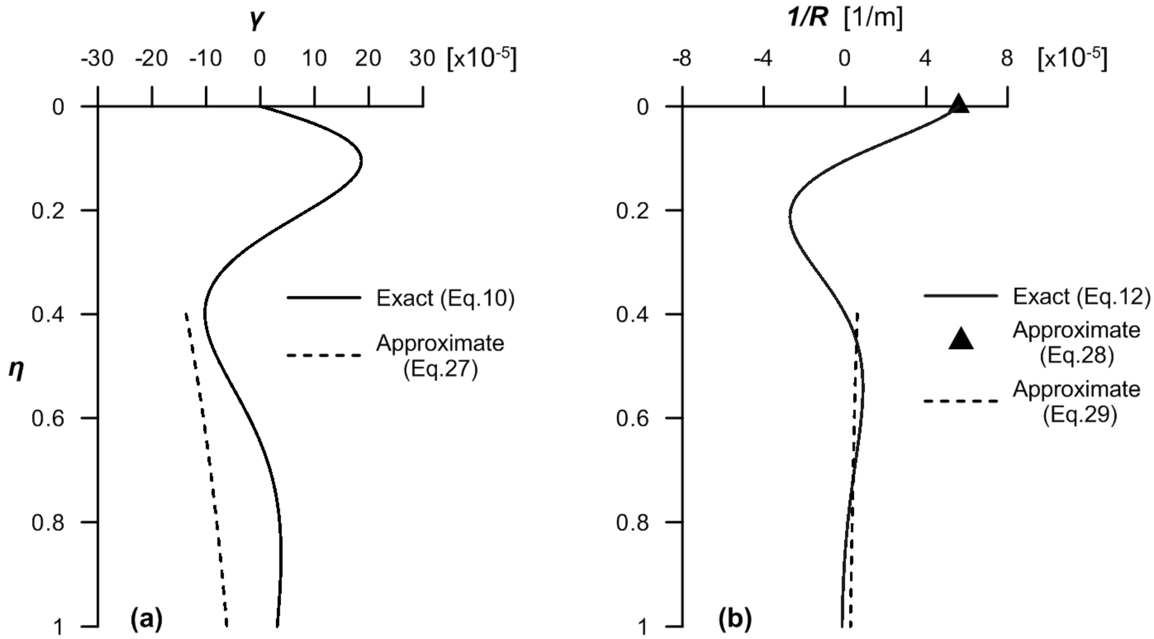

Figure S2: Comparison of shear strains (a) and curvatures (b) with depth, between the exact solution and the approximate solutions in Eqs. 27, 28 and 29 analysed for the "Profile B" case study; $\ddot{u}(0)$ = 1 m/s^2, $f = f_3$ = 5.13 Hz, ξ = 0.05

TABLES

Table S1. Soil layering and V_s values reported for the KiK-net station IBRH13 – "Profile A"

No	***Thickness (m)***	***Depth (m)***	***V_s (m/sec)***
1	1	1	170
2	15	16	280
3	8	24	400
4	10	34	600
5	10	44	1050
6	32	76	2600
7	----	-----	3000

Source: Data from NEID (2019).

Table S2. Data reported for the San Francisco Bay area profile – "Profile B"

No	***Thickness (m)***	***Depth (m)***	***V_s (m/sec)***
1	0.42	0.42	112
2	0.42	0.84	135
3	0.43	1.27	159
4	0.43	1.7	165
5	0.64	2.34	165
6	0.64	2.98	165
7	0.64	3.62	165
8	0.64	4.26	165
9	1.71	5.97	130
10	1.71	7.68	130
11	1.93	9.61	130
12	2.18	11.79	130
13	3.21	15	184
14	3.21	18.21	184
15	4.3	22.51	184
16	5	27.51	257
17	2	29.51	232
18	4	33.51	300
19	16.5	50	300
20	10	60	550

Source: Data from Gazetas and Dobry (1984).

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